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Khuri-Treiman-Type Equations for Three-Body Decay and Production Processes. II

R. PASQUIER
Institut de Physique Nucléaire, Division de Physique Théorique, 91 Orsay, France*

AND

J. Y. PASQUIER
Laboratoire de Physique Théorique et Hautes Energies, Faculté des Sciences, 91 Orsay, France*

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We discuss in a definite example the complications arising from the introduction of nonzero angular momenta in a dispersion model for three-body decay or production, based essentially on the elastic approximation in each subenergy channel. First we define production or decay amplitudes free from kinematical singularities. Then it is shown that a convenient choice of these amplitudes leads to integral equations to which the conclusions found in an earlier work in the zero-angular-momentum case apply for the most part, especially as regards three-body unitarity. Further details are given in the case of a three-pion final state.

I. INTRODUCTION

IN a previous work[1] (hereafter referred to as I), we have considered a model for three-body decay or production processes based on the elastic approximation in each subenergy channel. The study was restricted to the case of a spinless particle (or a spin $J=0$ production state) decaying into three final particles interacting pairwise in S-wave ($l=0$) states only.

Our present aim is to extend the discussion to the case of higher angular momenta. For definiteness we consider the simple case $J=1$, $l=1$, which is of great practical interest, especially in studying the three-pion system. Many notations and derivations have been already encountered in I; in the present paper, we just examine in more detail the new complications arising from the introduction of nonvanishing angular momenta. The model is here also named the Khuri-Treiman (KT) model.

The kinematical aspect of the problem is considered in Sec. II; our attention is particularly focused on the kinematical singularities that we have to get rid of before writing dispersion integrals. In Sec. III, we pass to the derivation of the integral equations themselves; kinematical singularities and problems of convergence lead us to work with reduced amplitudes to which the main considerations of the case $J=0$, $l=0$ apply, especially as concerns three-body unitarity. The required symmetry properties and the physical meaning of the kernels of the integral equations are investigated in Appendix B. In Appendix A are collected the kinematical formulas needed in the text. The three-pion case is examined with further details in Appendix C; in particular, isospin is introduced and the requirements of Bose-Einstein statistics are discussed.

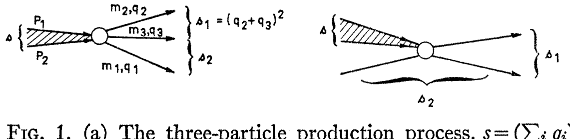

FIG. 1. (a) The three-particle production process. $s=(\sum_i q_i)^2$ is the square of the total three-body mass; $s_1=(q_2+q_3)^2$ is a final-state subenergy variable. (b) The process after particle (1) has been crossed. Both processes are also considered in the text as quasi-four-leg reactions in which $\sqrt{s}$ is the mass of a fictitious particle of spin J and helicity Λ_1.

II. KINEMATICS OF A PRODUCTION REACTION

For the sake of completeness, we recall here some general properties related to the partial-wave expansion of a production amplitude. Indeed many of the following derivations remain valid in the case of a decay with some convenient accommodations.

The production amplitude $\mathbf{R}(p,q)$ that we consider is illustrated in Fig. 1(a). p_μ ($\mu=1, 2$) and q_i ($i=1, 2, 3$) denote the four-momenta of the initial and final particles, respectively; for simplicity they are all assumed spinless and of mass unity. Such an amplitude depends upon two momentum-transfer invariants and three energy invariants which we choose to be the total energy variable in the c.m. system, $s=(\sum_\mu p_\mu)^2=(\sum_i q_i)^2$, and two of the subenergy variables $s_i=(q_j+q_k)^2$ related by $\sum_i s_i=s+3$.

The general form of the partial-wave expansion of $\mathbf{R}(p,q)$ in the total (or s) c.m. system reads[2,3]

$$\mathbf{R}(p,q)=\sum_J \sum_{\Lambda_i=-J}^{J} (2J+1) \times \mathfrak{D}_{\Lambda_i 0}{}^{J*}(\mathcal{E}^{-1}(\mathbf{q}_i)\mathcal{E}(\mathbf{p}_\mu))\mathbf{R}^{J\Lambda_i}(s,\{s_i\}), \quad (2.1)$$

where $\mathcal{E}(\mathbf{p}_\mu)$ and $\mathcal{E}(\mathbf{q}_i)$ stand for the rotations which transform a fixed given frame into body-fixed frames, which we also call $\mathcal{E}(\mathbf{p}_\mu)$ and $\mathcal{E}(\mathbf{q}_i)$, associated with the two initial collinear and three final coplanar momenta,

* Laboratoires associés au Centre National de la Recherche Scientifique.

[1] R. Pasquier and J. Y. Pasquier, Phys. Rev. **170**, 1294 (1968).

[2] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962).

[3] D. Branson, P. V. Landshoff, and J. C. Taylor, Phys. Rev. **132**, 902 (1963).

respectively. The notation specifies that $\mathbf{p}_\mu$ and $\mathbf{q}_i$ play a privileged role in the orientation of these frames. More explicitly, the z axis of $\mathcal{E}(\mathbf{p}_\mu)$ is chosen in the direction of $\mathbf{p}_\mu$ and the x-z plane is defined up to an arbitrary rotation around this momentum, which allows one to take the second subscript of the $\mathfrak{D}$ function[4] equal to zero.[3] As regards $\mathcal{E}(\mathbf{q}_i)$, two alternatives are generally considered. They correspond either to a choice of the z and y axes in the direction $-\mathbf{q}_i=\mathbf{q}_j+\mathbf{q}_k$, and $q_i\times q_j$, respectively,[2,3,5] or to a choice of the x and z axes in the direction $\mathbf{q}_i$ and $\mathbf{q}_i\times\mathbf{q}_j$, respectively.[3,5–7] [In both definitions (i,j,k) stands for a cyclic permutation of (1,2,3).] Notice that the first subscript Λ_i of $\mathfrak{D}$ in Eq. (2.1) always represents the projection of the angular momentum $\mathbf{J}$ along the z axis of $\mathcal{E}(\mathbf{q}_i)$.

The two preceding choices are referred to below as (h) and (t), respectively. Indeed we shall be essentially concerned with the choice (h). The choice (t) has been used elsewhere in other approaches of the three-body problem,[8] and for completeness we recall occasionally some obvious properties concerning it. With the choice (h), Eq. (2.1) reads, explicitly,

$$\mathbf{R}(p,q)=\sum_J\sum_{\Lambda_i=-J}^{J}(2J+1)\mathfrak{D}_{\Lambda_i0}{}^{J*}(\beta_i,\alpha_i,0)\times\mathbf{R}^{J\Lambda_i}(s,\{s_i\}),\quad(2.2)$$

where α_i and β_i are the polar and azimuthal angles of $\mathbf{p}_\mu$ in the frame $\mathcal{E}(\mathbf{q}_i)$, as shown in Fig. 2(a) for μ and i equal to 1.

Of course, depending on the choice of the privileged final momentum $\mathbf{q}_i$, different expansions of the type (2.1) [and thus (2.2)] are obtained. It is quite simple to compare the corresponding projected amplitudes, thanks to the orthogonality and group properties of the $\mathfrak{D}$ functions. We get in this way for i, $j=1$, 2, 3:

$$\mathbf{R}^{J\Lambda_i}=\sum_{\Lambda_j=-J}^{J}\mathfrak{D}_{\Lambda_j\Lambda_i}{}^{J*}(\mathcal{E}^{-1}(\mathbf{q}_j)\mathcal{E}(\mathbf{q}_i))\mathbf{R}^{J\Lambda_j}.\quad(2.3)$$

With the choice (h) for $\mathcal{E}(\mathbf{q}_i)$, the Euler angles associated with $\mathcal{E}^{-1}(\mathbf{q}_j)\mathcal{E}(\mathbf{q}_i)$ are $(0,\chi_{ji},0)$, where χ_{ji} is the oriented angle between $-\mathbf{q}_j$ and $-\mathbf{q}_i$ in the s c.m. system. Equation (2.3) then reads

$$\mathbf{R}^{J\Lambda_i}=\sum_{\Lambda_j=-J}^{J}d_{\Lambda_j\Lambda_i}{}^{J}(\chi_{ji})\mathbf{R}^{J\Lambda_j}.\quad(2.4)$$

With the choice (t) we have the diagonal relation

$$\mathbf{R}_t{}^{J\Lambda_i}=\sum_{\Lambda_j=-J}^{J}\delta_{\Lambda_i\Lambda_j}e^{i\Lambda_j\chi_{ji}}\mathbf{R}_t{}^{J\Lambda_j};\quad(2.5)$$

the subscript t means that we are dealing with amplitudes different from those of case (h); they are indeed simply linear combinations of them.

The summation over Λ_j in Eqs. (2.3) and (2.4) may be further reduced by taking account of parity conservation. As is well known,[5,6] the amplitudes $\mathbf{R}^{J\Lambda_i}$ of opposite Λ_i satisfy

$$\mathbf{R}^{J-\Lambda_i}=\eta(-)^{J+\Lambda_i}\mathbf{R}^{J\Lambda_i},\quad(2.6)$$

where η stands for the product of the parity of the initial state and of the intrinsic parities of the three final particles. This yields, in particular,

$$\mathbf{R}^{J0}=\eta(-)^J\mathbf{R}^{J0},$$

so that $\mathbf{R}^{J0}$ vanishes for J odd, $\eta=+1$, and for J even, $\eta=-1$. This allows us also to rewrite Eq. (2.4) as

$$\mathbf{R}^{J\Lambda_i}=\sum_{\Lambda_j=0}^{J}\epsilon_{\Lambda_j}e_{\Lambda_j\Lambda_i}{}^{J}(\chi_{ji})\mathbf{R}^{J\Lambda_j},\quad\Lambda_i\geqslant0,\quad(2.7)$$

where $\epsilon_0=1$, $\epsilon_m=2$ if $m=$ integer >0, and

$$e_{\Lambda_j\Lambda_i}{}^{J}(\chi)=\tfrac{1}{2}[d_{\Lambda_j\Lambda_i}{}^{J}(\chi)+\eta(-)^{J+\Lambda_i}d_{-\Lambda_j\Lambda_i}{}^{J}(\chi)].\quad(2.8)$$

Now, with the help of the preceding relations, it is possible to investigate the kinematical singularities of the partial-wave projections of $\mathbf{R}(p,q)$. For definiteness we consider $\mathbf{R}^{J\Lambda_i}$. The study may be carried out by

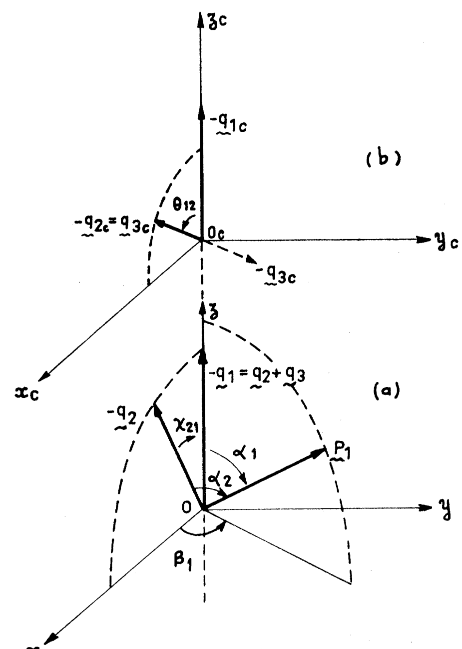

FIG. 2. The coordinate systems and vectors relevant to the angular decompositions of Secs. II and III. The present choice of axes is referred to as (h) in the text. Moreover, the vector $\mathbf{p}_\mu$ of Eq. (2.2) is here denoted $\mathbf{p}_1$, (a) is in the c.m. of the three final particles of Fig. 1(a). (b) is in the c.m. of particles (2) and (3) [see Fig. 1(b)].

[4] See, e.g., D. M. Brink and G. R. Satchler, *Angular Momentum* (Oxford University Press, New York, 1962).
[5] S. M. Berman and M. Jacob, Phys. Rev. **139**, B1023 (1965).
[6] J. Werle, Phys. Rev. Letters 4, 127 (1963); Nucl. Phys. **44**, 579 (1963).
[7] R. L. Omnès, Phys. Rev. **134**, B1358 (1964).
[8] J. L. Basdevant and R. E. Kreps, Phys. Rev. **141**, 1398 (1966); **141**, 1404 (1966).

copying a method used by Wang[9]: first, $\mathbf{R}^{J\Lambda_i}$ is expanded over the s_1 c.m. angular-momentum states. (The convergence of such an expansion within the KT model was briefly discussed in Appendix A of I.) The coordinate system relevant for the decomposition is shown in Fig. 2(b): The s_1 c.m. frame is simply deduced from $\mathcal{E}(\mathbf{q}_1)$ by a Lorentz transformation along $-\mathbf{q}_1$.[2] The result is

$$\mathbf{R}^{J\Lambda_1}(s,\{s_i\})=\sum_{l_1}(2l_1+1)d_{\Lambda_1 0}{}^{l_1}(\theta_{12})\mathbf{R}^{Jl_1\Lambda_1}(s,s_1)\,,\quad(2.9)$$

where θ_{12} is the oriented angle between the momenta of particles 1 and 2 in the s_1 c.m. system (cf. Appendix A). This shows that the kinematical singularities and zeros in s_2 (or s_3), which all arise from the $d_{\Lambda_1 0}{}^{l_1}(\theta_{12})$, can be isolated by setting

$$\mathbf{R}^{J\Lambda_1}(s,\{s_i\})=(\tfrac{1}{2}\sin\theta_{12})^{|\Lambda_1|}\bar{\mathbf{R}}^{J\Lambda_1}(s,\{s_i\})\,.\quad(2.10)$$

Then Eq. (2.7) may be written as

$$\bar{\mathbf{R}}^{J\Lambda_1}(s,\{s_i\})=\sum_{\Lambda_2=0}^{J}\bar{M}_{\Lambda_1\Lambda_2}\bar{\mathbf{R}}^{J\Lambda_2}(s,\{s_i\})\,,\qquad(2.11)$$

with

$$\bar{M}_{\Lambda_1\Lambda_2}=\epsilon_{\Lambda_2}(2/\sin\theta_{12})^{|\Lambda_1|}e_{\Lambda_2\Lambda_1}{}^{J}(\chi_{21})(\tfrac{1}{2}\sin\theta_{23})^{|\Lambda_2|}\,.\quad(2.12)$$

(The definition of θ_{23} follows from that of θ_{12} by a cyclic permutation over the indices.) Equation (2.11) implies that the kinematical singularities and zeros in s_1 of $\bar{\mathbf{R}}^{J\Lambda_1}$ are all in the explicitly known factor $\bar{M}_{\Lambda_1\Lambda_2}$, since $\bar{\mathbf{R}}^{J\Lambda_2}$ has none. We may thus define "regularized" amplitudes $\hat{\mathbf{R}}^{J\Lambda_1}$ through

$$\mathbf{R}^{J\Lambda_1}(s,\{s_i\})=\beta_{J\Lambda_1}{}^{(\eta)}\times\hat{\mathbf{R}}^{J\Lambda_1}(s,\{s_i\})\,,\qquad(2.13)$$

where the functions $\beta_{J\Lambda_1}{}^{(\eta)}$ may be deduced from Eqs. (2.10) and (2.12); their expressions are given in Appendix A for $J=1$, $\eta=\pm1$, (recall that $\mathbf{R}^{10}=0$ for $\eta=+1$).

Similar relations hold between any $\mathbf{R}^{J\Lambda_i}$ and $\hat{\mathbf{R}}^{J\Lambda_i}$; the associated $\beta_{J\Lambda_i}{}^{(\eta)}$ follow from $\beta_{J\Lambda_1}{}^{(\eta)}$ by a cyclic permutation over the indices. It is worth noticing the simplicity of such equations: All the kinematical analytic structure is contained in the factor β, whereas the remaining part $\hat{\mathbf{R}}$ just contains the dynamical information. Had we worked with the amplitudes $\mathbf{R}_t{}^{J\Lambda_i}$, it would not have been possible to separate the kinematical from the dynamical analytic structure in such a simple way. Notice also that, as a consequence of Eqs. (2.7) and (2.13), the $\hat{\mathbf{R}}^{J\Lambda_i}(s,\{s_i\})$ are related by

$$\hat{\mathbf{R}}^{J\Lambda_i}(s,\{s_i\})=\sum_{\Lambda_j=0}^{J}\hat{M}_{\Lambda_i\Lambda_j}\hat{\mathbf{R}}^{J\Lambda_j}(s,\{s_i\})\,,\qquad(2.14)$$

where

$$\hat{M}_{\Lambda_i\Lambda_j}=(1/\beta_{J\Lambda_i}{}^{(\eta)})\epsilon_{\Lambda_j}e_{\Lambda_i\Lambda_j}{}^{J}(\chi_{ji})\beta_{J\Lambda_j}{}^{(\eta)}\,.\quad(2.15)$$

[9] L. L. C. Wang, Phys. Rev. 142, 1187 (1965).

Now, what remains to be done is to look at an eventuality that we have forgotten for a time: It may happen[10,11] that the matrix $\bar{M}$ in Eq. (2.11) [or $\hat{M}$ in Eq. (2.14)] is degenerate for particular values of the variables (we do not consider the case in which these matrices reduce to one element, as for $J=1$, $\eta=+1$). Then its different rows no longer remain independent, and suitable combinations of the $\mathbf{R}^{J\Lambda_i}$ ($\hat{\mathbf{R}}^{J\Lambda_i}$) must vanish at these points. The study of such kinematical constraints is easier in the present context by returning to Eq. (2.2), instead of analyzing the zeros of the determinant of $\bar{M}$ (or $\hat{M}$); if we restrict ourselves to the $J=1$, $\eta=-1$ case, this equation reads for $i=2$, 1[12]

$$\mathbf{R}=\cos\alpha_2\,\beta_{J0(2)}{}^{(-)}\hat{\mathbf{R}}^{10(2)}-\sqrt{2}\,\cos\beta_2\,\sin\alpha_2\,\beta_{J1(2)}{}^{(-)}\hat{\mathbf{R}}^{11(2)}\,,\quad(2.16)$$

$$=\cos\alpha_1\,\beta_{J0(1)}{}^{(-)}\hat{\mathbf{R}}^{10(1)}-\sqrt{2}\,\cos\beta_1\,\sin\alpha_1\,\beta_{J1(1)}{}^{(-)}\hat{\mathbf{R}}^{11(1)}\,.\quad(2.17)$$

The first expression tells us that $\mathbf{R}$ has no kinematical singularity in s_1 since each term has none (see Appendix A). As a consequence, the combination (2.17) must be also regular in s_1, although each term indeed has poles at $s_1=(\sqrt{s}\pm1)^2$, as follows from the values of $\beta_{J\Lambda_1}{}^{(-)}$ and of the trigonometrical lines of α_1 and β_1 (see Appendix A); this implies that

$$2(\sqrt{s})\hat{\mathbf{R}}^{10(1)}+2N_\chi(s_1,s_2)\hat{\mathbf{R}}^{11(1)}=O(k_1{}^2)\quad(2.18)$$

at $s_1=(\sqrt{s}\pm1)^2$ (see Appendix A for the definition of N_χ). Correspondingly, one of the $\mathbf{R}_t{}^{1\Lambda_1}$ [remember the diagonal relation Eq. (2.5)] has a zero at the same points. At this stage, these conditions just express the fact that different projections of a given amplitude $\mathbf{R}$ cannot take on independent values at some special points. A more usual interpretation of them will be encountered below.

III. DYNAMICAL MODEL

We are now in a position to extend to the case $J\neq0$ the basic assumptions of the dynamical model presented in I. Indeed, the regularized amplitudes $\hat{\mathbf{R}}^{J\Lambda_i}(s,\{s_i\})$ of Sec. II [see Eq. (2.13)] just possess dynamical cuts and appear as the equivalent of the decay amplitude $\mathcal{R}(s,\{s_i\})$ considered in the case $J=0$, $l=0$ [the latter is nothing but $\hat{\mathbf{R}}^{00}(s,\{s_i\})$]. Their analytic structure is also assumed only to take account of the elastic cut in each subchannel s_i of the five-leg process in Fig. 1; also, only a finite number of angular momenta is retained in the expression of the associated discontinuities.

As in I, such amplitudes are expected to represent at the same time three scattering reactions as in Fig. 1(b)

[10] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) 46, 239 (1968).

[11] S. Frautschi and L. Jones, Phys. Rev. 164, 1918 (1967).

[12] The indices between brackets recall the index of Λ, i.e., the privileged axis of projection.

and a production (or decay) reaction as in Fig. 1(a), thanks to convenient analytic continuations; recall that for $s \geqslant 9$, the physical regions of all these reactions are bound by the same cubic curve[13] associated with the quasi-four-leg processes of Fig. 1 with $m=\sqrt{s}$ as the mass of a fictitious particle. This assumption allows us to evaluate the two-body discontinuities of these amplitudes for values of the invariants in the physical region of the "scattering" process [Fig. 1(b)]. The integral equations are also first written in the same region and then analytically continued up to the decay (or production)[14] region [Fig. 1(a)].

It happens that in such continuations the relations (2.14) and (2.18) which we shall impose between the $\hat{\mathbf{R}}^{J\Lambda_i}$ remain unchanged, since the matrix $\hat{M}$ is free from branch point singularities. An important feature is that in the scattering region these relations are identical with the usual "crossing" relations[9,10,15] and the kinematical constraints[10,16] considered for the regularized helicity amplitudes. This means that one may associate a helicity amplitude with any $\hat{\mathbf{R}}^{J\Lambda_i}$ and consider it as the analytic continuation of the projected amplitude $\mathbf{R}^{J\Lambda_i}$ defined in Sec. II; the $\mathbf{R}_t^{J\Lambda_i}$ which are linear combinations of the $\mathbf{R}^{J\Lambda_i}$ would then correspond in the same way to transversity amplitudes.[10] Of course, such a correspondence[17] is already implicit in Sec. II, since almost all the methods used to exhibit kinematical singularities in the decay region are the analogs of well-known methods used for $2\rightarrow 2$ helicity amplitudes.[9,10] These remarks also agree with an assumption which we could have formulated at the beginning: The development (2.1) derived in the decay region may as well be obtained by a two-step procedure as follows:

(a) First, the five-leg amplitude $\mathbf{R}$ is considered under the aspect of Fig. 1(b) and expanded over the angular-momentum states of the subenergy $\sqrt{s}$. As noted elsewhere,[2] with the choice (h) of axes considered in Sec. II this expansion involves amplitudes which have the same properties as the helicity amplitudes associated with a scatteringlike reaction, also illustrated in Fig. 1(b).

(b) Then the result is analytically continued up to decaying values of $\sqrt{s}$.

Let us now examine the implications of the model in detail. First, it is worth mentioning that the elastic approximation and the limitation of the number of partial waves essentially reduce the amplitude resulting from the model to a sum of three components, each of them corresponding to a subchannel s_i and possessing just the associated elastic cut. More explicitly, we shall always have

$$\hat{\mathbf{R}}^{J\Lambda_i}=\hat{\mathbf{R}}_0^{J\Lambda_i}+\hat{\mathbf{R}}_1^{J\Lambda_i}+\hat{\mathbf{R}}_2^{J\Lambda_i}+\hat{\mathbf{R}}_3^{J\Lambda_i}; \qquad (3.1)$$

thus from Eqs. (2.13) and (2.2):

$$\mathbf{R}^{J\Lambda_i}=\mathbf{R}_0^{J\Lambda_i}+\mathbf{R}_1^{J\Lambda_i}+\mathbf{R}_2^{J\Lambda_i}+\mathbf{R}_3^{J\Lambda_i}, \qquad (3.2)$$

$$\mathbf{R}=\mathbf{R}_0+\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3, \qquad (3.2')$$

with obvious notations. In Eq. (3.1) $\hat{\mathbf{R}}_0^{J\Lambda_i}$ stands for processes not induced by two-body interactions, while $\hat{\mathbf{R}}_j^{J\Lambda_i}$ has to be understood as the contribution of the s_j to the s_i two-body channels.

However, at this stage, we know about the $\hat{\mathbf{R}}_j^{J\Lambda_i}$ only that they possess the elastic s_j cut with the same discontinuity as $\mathbf{R}^{J\Lambda_i}$ itself. As shown in I for $J=0$, if only the subenergy partial wave $l=0$ is retained in two-body unitarity, this input alone is sufficient to build up a model which satisfies some convenient properties:

(a) The integral equations of the model converge under conditions of practical interest and involve the unknown functions under only one integration.

(b) As in the usual isobaric models,[18] the angular dependences [i.e., the $\mathfrak{D}^J$ and the d^l functions of Eqs. (2.2) and (2.9)—the second is indeed trivial if $l=0$] appear explicitly in the resulting amplitude.

(c) Three-body unitarity can be nearly satisfied under conditions of practical interest.

On the other hand, for $l\neq 0$, many ways appear possible when one wants to reconstruct the amplitudes from their given discontinuities. Therefore, these only generate an equivalence class from which we have to pick the most appropriate amplitudes. A natural way to proceed is to consider the above properties (a)–(c) of the case $l=0$ as conditions which must hold if $l\neq 0$. Of course, we also require the amplitudes $\hat{\mathbf{R}}^{J\Lambda_i}$ resulting from the model to be effectively the regularized projections of a decay amplitude and thus to satisfy the kinematical relations (essentially crossing and constraints) derived in Sec. II. Since $\hat{\mathbf{R}}^{J\Lambda_i}$ has indeed the form (3.1), one can fulfill this condition—referred to below as (d)—by imposing it on each term on the right-hand side of Eq. (3.1). By doing so, it is then possible to work with only a restricted number of functions $\hat{\mathbf{R}}_j^{J\Lambda_i}$ and define the others by Eq. (2.14); we choose here the diagonal amplitudes $\hat{\mathbf{R}}_i^{J\Lambda_i}$ as the basic ones, and for definiteness specialize to $\hat{\mathbf{R}}_1^{J\Lambda_1}$.

The expression for the two-body discontinuity of such an amplitude is obtained by applying two-body uni-

[13] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

[14] From now on, we use the word "decay" for decay or production.

[15] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[16] H. F. Jones, Nuovo Cimento **50**, 814 (1967).

[17] Y. Kim, Phys. Rev. **125**, 1771 (1962) attempted to relate decay and scattering amplitudes in the nonzero-spin case by working with spinor invariant amplitudes. These, and the helicity (or transversity) amplitudes that we consider in the present work, appear thus to provide two sets of amplitudes which can be used in decay problems for particles with spin, just as in the scattering problems (Refs. 15 and 16).

[18] S. J. Lindenbaum and R. M. Sternheimer, Phys. Rev. **105**, 1874 (1957); M. Olsson and G. B. Yodh, Phys. Rev. Letters **10**, 353 (1963); B. Deler and G. Valladas, Nuovo Cimento **45**, A559 (1966) (this work contains further references).

tarity in the s_1 c.m. system of the process in Fig. 1(b). This reads, formally,[19]

$$\mathrm{disc}\langle q_1p_1p_2|\mathbf{R}|q_2q_3\rangle = 2i\sum_2 \langle q_1p_1p_2|\mathbf{R}|q_2'q_3'\rangle \times \langle q_2'q_3'|\mathrm{M}^\dagger|q_2q_3\rangle, \quad (3.3)$$

where M is the $2\to 2$ scattering amplitude between particles (2) and (3), and $\sum_2$ stands for the two-body phase-space integration over q_2' and q_3' with nondistorted contours for small s.

Such an equation can be reduced by expanding each side in partial waves[2] (the relevant coordinate systems are shown in Fig. 2). On the left-hand side it is convenient to use an expansion in the angular-momentum states involved, in the c.m. system of the subenergy s; this is the analytic continuation to small s values of the expansion (2.2) with $i=1$. Notice that the trigonometrical lines of α_1 and β_1 involved in this expansion are free from singularities at $s_1=4$ (see Appendix A). On the right-hand side, the two-body amplitude M may as usual be expanded over Legendre polynomials. As regards $\langle q_1p_1p_2|\mathbf{R}|q_2'q_3'\rangle$, it is convenient to expand it over the angular-momentum states of both the s and s_1 c.m. system [the result looks like the combination of Eqs. (2.2) and (2.9) with the roles of l and J exchanged]. Thanks to all these decompositions, the phase-space integration $\sum_2$ in Eq. (3.3) is easy to carry out; after some straightforward manipulations we obtain

$$\mathrm{disc}\hat{\mathrm{R}}_1{}^{J\Lambda_1}(s,s_1,s_2) = \mathrm{disc}\hat{\mathbf{R}}^{J\Lambda_1}(s,\{s_i\}) = 2i\rho(s_1)\sum_{l_1}(2l_1+1)\mathrm{M}^{(l_1)*}(s_1) \times \mathbf{R}^{Jl_1\Lambda_1}(s,s_1)\frac{d_{\Lambda_1 0}{}^{l_1}(\theta_{12})}{\beta_{J\Lambda_1}{}^{(\eta)}(s_1,s_2)}, \quad (3.4)$$

with $\rho(s_1)=K_1/s_1$. K_1 and some trigonometrical lines of θ_{12} are given in Appendix A. $\mathrm{M}^{(l_1)}$ stands for the l_1-wave projection of M in the s_1 c.m. system, and $\mathbf{R}^{Jl_1\Lambda_1}(s,s_1)$ appears as the double projection of $\mathbf{R}$ over both the J and l angular-momentum states (it is the same whatever the order of the projections). From Eq. (2.9) we have, in particular,

$$\mathbf{R}^{Jl_1\Lambda_1}(s,s_1) = \tfrac{1}{2}\int_{-1}^{+1}\mathbf{R}^{J\Lambda_1}(s,s_1,s_2')d_{\Lambda_1 0}{}^{l_1}(\theta_{12}')d\cos\theta_{12}', \quad (3.5\mathrm{a})$$

$$\mathbf{R}^{Jl_1\Lambda_1}(s,s_1) = \mathbf{R}_0{}^{Jl_1\Lambda_1} + \mathrm{R}_1{}^{Jl_1\Lambda_1}(s,s_1) + \sum_{\Lambda_2\geqslant 0}\tfrac{1}{2}\int_{-1}^{+1}\epsilon_{\Lambda_2}e_{\Lambda_2\Lambda_1}{}^{J}(\chi_{21}')\beta_{J\Lambda_2}{}^{(\eta)}(s_2',s_3')\hat{\mathrm{R}}_2{}^{J\Lambda_2}(s,s_2',s_3')d_{\Lambda_1 0}{}^{l_1}(\theta_{12}')d\cos\theta_{12}'$$
$$+\sum_{\Lambda_3\geqslant 0}\tfrac{1}{2}\int_{-1}^{+1}\epsilon_{\Lambda_3}e_{\Lambda_3\Lambda_1}{}^{J}(\chi_{31}')\beta_{J\Lambda_3}{}^{(\eta)}(s_3',s_1)\hat{\mathrm{R}}_3{}^{J\Lambda_3}(s,s_3',s_1)d_{\Lambda_1 0}{}^{l_1}(\theta_{12}')d\cos\theta_{12}'. \quad (3.5\mathrm{b})$$

The last equation follows from the representation (3.2) and the crossing property, Eq. (2.13). The primes on the χ's and the subenergies ($s_2'+s_3'+s_1=s+3$) specify that they are functions of $\cos\theta_{12}'$.

Indeed, in the equal-mass symmetric case which we consider, Eq. (3.5′) may be simplified. To show this, one can choose s_2' and s_3' as new variables in the first and second integrals of the equation and express the functions $\beta_{J\Lambda_i}{}^{(\eta)}\hat{\mathrm{R}}_i{}^{J\Lambda_i}\equiv\mathrm{R}_i{}^{J\Lambda_i}$, $i=2$, 3, in terms of the partial-wave amplitudes $\mathrm{R}_i{}^{Jl_i\Lambda_i}(s,s_i)$ [see Eq. (2.9)]. Because these amplitudes, for the same values of l and Λ, may be considered as the same function whatever the choice of i is (this assumption is further examined in Appendix C), the two integrands appear as the same function, one of s_2', the other of s_3', to be integrated over the same range of values. In the end, the contributions of channels (2) and (3) to Eq. (3.5′) are thus identical.

Now, as mentioned above, different ways are open to build up the amplitude $\hat{\mathrm{R}}_1{}^{J\Lambda_1}$ from the two-body discontinuity equation (3.4), in which we keep by now only the wave $l_1=1$. The more natural way is to write the dispersion relation for $\hat{\mathrm{R}}_1{}^{J\Lambda_1}$ itself. However, by doing so, the kinematical factor $d_{\Lambda_1 0}{}^{1}(\theta_{12})/\beta_{J\Lambda_1}{}^{(\eta)}(s_1',s_2)$ of Eq. (3.4) remains under the integral and this leads to two main difficulties:

(1) It is not always possible to put the resulting integral equations into a single variable form because s_2 and the dispersive variable s_1' can mix together in this factor (this is a polynomial of degree 0 or 1 in s_2).

(2) The integral equations do not converge rapidly (this is a well-known difficulty of Cini-Fubini approximations[20]).

Both these drawbacks may be partially removed by splitting the dispersion integral into two parts: one in which the rational part (this alone depends on s_2) of $d_{\Lambda_1 0}{}^{1}(\theta_{12})/\beta_{J\Lambda_1}{}^{(\eta)}(s_1',s_2)$ is put outside the integral, the other which does not possess the cut $s_1\geqslant 4$ and can thus be added to the inhomogeneous term. This procedure is indeed equivalent to retaining only the wave[21] $l=1$ in

[19] In this form of two-body unitarity, the two-body squared energy variables in M and R are taken below and above the two-body cuts, respectively. One may write another form in which these positions are exchanged. As noticed elsewhere (Refs. 24 and 37), both expressions are equivalent on the "principal" sheet of the discontinuity.

[20] See, e.g., S. C. Frautschi, *Regge poles and S-matrix Theory* (W. A. Benjamin, Inc., New York, 1963).

[21] In the present model, as one can verify, this is the only partial-wave amplitude in $\hat{\mathrm{R}}_1{}^{J\Lambda_1}$ which possesses the cut $s_1\geqslant 4$.

$\hat{R}_1{}^{J\Lambda_1}(s,s_1,s_2)$, or to writing the dispersion integral for $\hat{R}_1{}^{J\Lambda_1}$ with two subtractions at $s_1=0$ and $s_1=\frac{1}{2}(s+3-s_2)$ if $\Lambda_1=0$. The same result also occurs in writing the dispersion integrals for the functions $g_1{}^{J\Lambda_1}(s_1)$ defined through

$$\hat{R}_1{}^{J\Lambda_1}(s,s_1,s_2)=\xi_{\Lambda_1}(s_1,s_2)g_1{}^{J\Lambda_1}(s,s_1)\,,\qquad(3.6)$$

where[12] $\xi_{1(1)}=1$ and $\xi_{0(1)}=s_1\times\gamma(s_1,s_2)$ (γ is defined in Appendix A). The two-body discontinuity of these amplitudes $g_1{}^{J\Lambda_1}$ can be derived by combining Eqs. (3.4), (3.5), and (3.6). In the case which we consider ($J=1$, three identical final bosons, so that $g_i{}^{J\Lambda_i}=g^{J\Lambda}$ for any i), we can write them easily. If for simplicity we drop the index J on $g^{J\Lambda}$ and, except on the limits of the integrals, the variable s, we obtain

$\eta=+1$:

$$\mathrm{disc}g^1(s_1)=2i\rho(s_1)\mathrm{M}^{(1)*}(s_1)\left(g^1(s_1)+2\times3\times\frac{2s_1{}^2}{k_1{}^3K_1{}^3}\int_{s_-(s,s_1)}^{s_+(s,s_1)}\varphi(s_1,s_2')g^1(s_2')ds_2'\right),\qquad(3.7)$$

$\eta=-1$:

$$\mathrm{disc}g^0(s_1)=2i\rho(s_1)\mathrm{M}^{(1)*}(s_1)$$
$$\times\left(g^0(s_1)+2\times3\times\frac{4s_1{}^2}{k_1{}^3K_1{}^3}\int_{s_-(s,s_1)}^{s_+(s,s_1)}\gamma(s_1s_2')\frac{-N_\chi(s_1,s_2')\gamma(s_2',s_1)s_2'g^0(s_2')+2(2s)^{1/2}\varphi(s_1,s_2')g^1(s_2')}{k_2'^2}ds_2'\right),\qquad(3.8)$$

$$\mathrm{disc}g^1(s_1)=2i\rho(s_1)\mathrm{M}^{(1)*}(s_1)$$
$$\times\left(g^1(s_1)+2\times3\times\frac{2s_1{}^2}{k_1{}^3K_1{}^3}\int_{s_-(s,s_1)}^{s_+(s,s_1)}\varphi(s_1,s_2')\frac{(2s)^{1/2}\gamma(s_2',s_1)s_2'g^0(s_2')+N_\chi(s_1,s_2')g^1(s_2')}{k_2'^2}ds_2'\right),\qquad(3.9)$$

where the $s_\pm(s,s_1)$ and the meaning of the s_2' paths of integration are the same as in I; the functions k_i, K_i, φ, N_χ, and γ are given in Appendix A.

As in I, we can make the following remarks on the integral equations which can be built up from these discontinuities:

(1) These integral equations can be simplified by means of an Omnès inversion.[22,23] This introduces the N and D functions of the two-body scattering amplitude $\mathrm{M}^{(1)}=\mathrm{N/D}$ and a natural function to work with appears to be $f^\Lambda=g^\Lambda\mathrm{D}$.

(2) The analytic continuation of these equations from small [Fig. 1(b)] to decay values of s [Fig. 1(a)] needs the consideration of the "principal" sheet[24] of the two-body discontinuity: recall that by definition, the phase-space contour remains undistorted on this sheet; correspondingly, the two-body discontinuity is free from non-Landau singularity[25] at $s_1=(\sqrt{s}-1)^2$, whereas it would behave like $k_1{}^{-3}$ (recall that $l=1$) on a nonprincipal sheet.

(3) The exchange of the order of integrations leads to a single-variable representation (S.V.R.) of the amplitude.[23]

This last property is nothing but the condition previously referred to as (a) (at this stage we assume that the factors N and D insure the convergence of the integral equations). Besides, condition (b) is just a consequence of the definition of f^Λ and g^Λ. It remains to examine conditions (c) and (d).

By construction, condition (d) is in part automatically satisfied: From the amplitudes f^Λ and g^Λ which result from our integral equations we can reconstruct, successively, $\hat{R}_i{}^{J\Lambda_i}$, $\hat{R}_j{}^{J\Lambda_i}$ [Eq. (2.14)], $\hat{\mathbf{R}}^{J\Lambda_i}$ [Eq. (3.1)], and $\mathbf{R}^{J\Lambda_i}$ [Eq. (2.13)]; the coefficients which appear in these combinations are generally free from any singularity at $s_i=4$ so that the resulting amplitudes have the correct two-body discontinuities and satisfy the crossing relations of Sec. II. Thus only kinematical constraints require more attention. In terms of the f^Λ (or g^Λ) these read [see Eq. (2.18)]

$$(\sqrt{s}+\epsilon)f^0((\sqrt{s}+\epsilon)^2)+\sqrt{2}f^1((\sqrt{s}+\epsilon)^2)=0\,,\qquad\epsilon=\pm1\,.\qquad(3.10)$$

As one can verify, the two-body discontinuities of the f^Λ automatically satisfy these relations, but there is *a priori* no reason why the same property holds for the reconstructed amplitudes.

It is possible, nevertheless, to build up amplitudes which do fulfill this condition by first writing the Cauchy integral for the functions

$$\frac{(\sqrt{s}+\epsilon)f^0(s_1)+\sqrt{2}f^1(s_1)}{s_1-(\sqrt{s}+\epsilon)^2}\,,\qquad(3.11)$$

which must be regular at $s_1=(\sqrt{s}+\epsilon)^2$, and then recombining the results. The amplitudes f^Λ that one obtains in this way automatically satisfy the condition (3.10) if the inhomogeneous terms are assumed to do so.

[22] R. L. Omnès, Nuovo Cimento **8**, 1244 (1958).
[23] I. J. R. Aitchison, Phys. Rev. **137**, B1070 (1965).
[24] R. C. Hwa, Phys. Rev. **134**, B1086 (1964). See also Appendix A of I, which contains further references.
[25] D. B. Fairlie, P. V. Landshoff, J. Nuttall, and J. C. Polkinghorne, J. Math. Phys. **3**, 594 (1962).

Their S.V.R. read finally for $\eta=-1$:

$$
\begin{aligned}
f^0(s_1)&=f_0{}^0(s_1)+2\int_{-\infty}^{(\sqrt{s}-1)^2} ds_2'\ {}^0\mathrm{X}^0(s_2',s,s_1)\frac{f^0(s_2')}{\mathrm{D}(s_2')}\\
&\quad+2\int_{-\infty}^{(\sqrt{s}-1)^2} ds_2'\ {}^1\mathrm{X}^0(s_2',s,s_1)\frac{f^1(s_2')}{\mathrm{D}(s_2')},\\
f^1(s_1)&=f_0{}^1(s_1)+2\int_{-\infty}^{(\sqrt{s}-1)^2} ds_2'\ {}^0\mathrm{X}^1(s_2',s,s_1)\frac{f^0(s_2')}{\mathrm{D}(s_2')}\\
&\quad+2\int_{-\infty}^{(\sqrt{s}-1)^2} ds_2'\ {}^1\mathrm{X}^1(s_2',s,s_1)\frac{f^1(s_2')}{\mathrm{D}(s_2')},
\end{aligned}
\tag{3.12}
$$

where the kernels

$$
{}^{\Lambda_2}\mathrm{X}^{\Lambda_1}(s_2',s,s_1)=-\theta(-s_2')\ {}^{\Lambda_2}\mathrm{K}^{\Lambda_1}(s_2',s,s_1)+\theta(s_2')\ {}^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)
\tag{3.13}
$$

satisfy the same kinematical constraints in s_1 as the $f^{\Lambda_1}(s_1)$.

Similarly for $\eta=+1$, we have

$$
f^1(s_1)=f_0{}^1(s_1)+2\times\int_{-\infty}^{(\sqrt{s}-1)^2} ds_2'\ {}^1\mathrm{X}^1(s_2',s,s_1)\frac{f^1(s_2')}{\mathrm{D}(s_2')},
\tag{3.14}
$$

with

$$
{}^1\mathrm{X}^1(s_2',s,s_1)=-\theta(-s_2')\ {}^1\mathrm{K}^1(s_2',s,s_1)+\theta(s_2')\ {}^1\Delta^1(s_2',s,s_1)\,.
\tag{3.15}
$$

The kernels ${}^{\Lambda_2}\mathrm{K}^{\Lambda_1}$ and ${}^{\Lambda_2}\Delta^{\Lambda_1}$ in these equations play the same roles as the K and $\Delta^{(1)}$ in I. Their full expression is here rather complicated and given in Appendix B.[26]

It is interesting to notice that for $\eta=-1$ the subtractions at $s_1=(\sqrt{s}+\epsilon)^2$ have by the way increased the convergence of the integral equations [such a property of the combination (3.11) has already been pointed out by Jones[16] in another problem]. When N reduces to the centrifugal factor $q_{2c}{}^2=K_1{}^2/4s_1$, some kernels ${}^{\Lambda_2}\mathrm{K}^{\Lambda_1}$ become indeed only meaningful once the subtractions are done.

From Eqs. (3.12)-(3.15) one may evaluate the discontinuity of the amplitude f^Λ across the cut $s\geqslant 9$, and then generate reduced $3\to 3$ amplitudes ${}^{\Pi}\bar{\psi}^{\Lambda}$ in the manner recalled in I [cf. (B8) and (B9) in I]. The integral equations they satisfy have here again as inhomogeneous terms the ${}^{\Pi}\Delta^{\Lambda}$, and have the same kernels as the integral equations for the f^Λ. The associated amplitudes ${}^{\Pi}\Psi^{\Lambda}$ [cf. Eq. (4.4) in I] are

$\eta=+1$:

$$
{}^1\Psi^1(s_2',s,s_1)=\tfrac{3}{4}s\frac{n(s_2')}{k_2'^3\mathrm{D}(s_2')}\,{}^1\bar{\psi}^1(s_2',s,s_1)\frac{1}{\mathrm{D}(s_1)},
$$

$\eta=-1$:

$$
{}^1\Psi^1(s_2',s,s_1)=\tfrac{3}{4}s\frac{n(s_2')}{k_2'\mathrm{D}(s_2')}\,{}^1\bar{\psi}^1(s_2',s,s_1)\frac{1}{\mathrm{D}(s_1)},
$$

$$
{}^1\Psi^0(s_2',s,s_1)=\tfrac{3}{4}s\frac{n(s_2')}{k_2'\mathrm{D}(s_2')}\,{}^1\bar{\psi}^0(s_2',s,s_1)\frac{1}{\mathrm{D}(s_1)},
$$

$$
{}^0\Psi^1(s_2',s,s_1)=\frac{3s}{s_2'}\frac{n(s_2')}{k_2'\mathrm{D}(s_2')}\,{}^0\bar{\psi}^1(s_2',s,s_1)\frac{1}{\mathrm{D}(s_1)},
$$

$$
{}^0\Psi^0(s_2',s,s_1)=\frac{3s}{s_2'}\frac{n(s_2')}{k_2'\mathrm{D}(s_2')}\,{}^0\bar{\psi}^0(s_2',s,s_1)\frac{1}{\mathrm{D}(s_1)}\,.
$$

These amplitudes satisfy integral equations whose inhomogeneous terms are simply related to the functions ${}^{\Pi}G^{\Lambda}(s_2',s,s_1)/\mathrm{D}(s_2')\mathrm{D}(s_1)$ considered in Appendix B [they differ by the additional terms considered there which remove the eventual poles of ${}^{\Pi}G^{\Lambda}(s_2',s,s_1)$ in $s_1=0$ and/or $s_2'=0$]. From the ${}^{\Pi}\Psi^{\Lambda}$ one can generate amplitudes ${}^{\Pi_\lambda}{}_\lambda\Psi_i{}^{\Lambda_i}(s_\lambda',s,s_i)$ as done in I [see in particular Eqs. (B9), (B11), and (B12) in paper I; in the present case the superscripts Π and Λ give rise to extra but nonessential complications]. Finally the full $3\to 3$ amplitude may be expressed as a linear combination of these ${}^{\Pi_\lambda}{}_\lambda\Psi_i{}^{\Lambda_i}$. The coefficients are nothing but the ξ of Eq. (3.6), the β of Eq. (2.13), the crossing matrices of Eq. (2.7), and the $\mathfrak{D}$ functions of Eq. (2.2). One can verify that all the discontinuities of this amplitude have just the form (if not the symmetry) expected from three-body unitarity.[27]

Finally, the examination of condition (c) leads us, as in I, to investigate the symmetry properties of the kernels ${}^{\Lambda_2}\mathrm{X}^{\Lambda_1}$, and in particular of the main part ${}^{\Lambda_2}\Delta^{\Lambda_1}$. It is shown in Appendix B that, despite their apparent complexity, for $\mathrm{N}=K_1{}^2/4s_1$, the ${}^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)$ are polar kernels, i.e., are symmetric with respect to s_2' and s_1 if multiplied by convenient functions of s_2' *or* s_1 (more precisely, they constitute a "symmetric" matrix in the coupled case $\eta=-1$). This situation is rather the same as that of the case $J=0$, $l=0$ and leads to the same two possible alternatives: to work with full or with truncated KT equations. But in the present case, for $\eta=-1$, new choices of truncated equations appear to be possible.[28] Indeed the ${}^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)$ only differ in this case from the $J=1$ projection of well-defined one-particle exchange (OPE) processes by terms correctly satisfying the kinematical constraints of Sec. II and

[26] In the expressions given in Appendix B, it appears that some kernels contain a factor $\sqrt{s}$. This can be removed by defining new amplitudes f^Λ free from kinematical singularities in s. The procedure is the same as that followed in Sec. II, with only the roles of $l=1$ and $J=1$ exchanged.

[27] G. N. Fleming, Phys. Rev. **135**, B551 (1964).

[28] This ambiguity results from the choice $l_1\neq 0$ and already occurs in the simpler case $J=0$, $l_1=1$ [considered, e.g., by M. O. Taha, Nuovo Cimento **42**, 201 (1966)].

having good symmetry properties. Two ways are then open: to include or not include these supplementary terms into the kernels of the truncated equations; in both cases, three-body unitarity is rigorously satisfied, at least in the decay region, if $N/q_{2c}^2=n=$const. As one can verify, these two new alternatives lead to different behaviors of $\hat{R}_1^{J0}$ at $s_1=0$: in the first case $\hat{R}_1^{J0}$ behaves like a constant, while in the second it behaves like s_1.

As in I, we can also try to investigate the effects of the singularities of the two-body forces by including nonconstant functions $n(s_1)$. Of course, it would be interesting to test numerically these different alternatives and to compare their respective predictions as concerns some physical quantities such as the masses and widths of three-body dynamical resonances induced by two-body resonant states, the Dalitz plot repartitions etc. For this purpose, the three-pion system is studied with further details in Appendix C. We especially introduce isospin and examine Bose-Einstein requirements. The approximation $l=1$ leads us to consider only the quantum numbers of the pion-pion ρ resonance in the two-body interaction. Such an approximation is of great practical interest, and has already been considered elsewhere in the context of Faddeev equations.[8]

IV. CONCLUSION

In I and in the present paper, we have discussed in particular cases of angular momenta an approach of the three-body relativistic problem based only on standard mass-shell S-matrix concepts and methods (analyticity in simply cut planes for small values of the external masses, analytic continuation with respect to these masses; and dispersion relations with eventual subtractions). Two-body unitarity is the essential input of such an approach. Correspondingly, the amplitude is built up by dispersing in the subenergy instead of the total energy variable. This can be a guarantee of simplicity since the associated discontinuities involve two particles instead of three.

In a first step, one may take account only of a finite number of subenergy partial waves. The initially double integral equation may then be put into a single variable integral form by an inversion of the order of integration, as studied in I. In this form the inhomogeneous term represents only the well-known Watson final-state formula,[29] while the integral term represents the corrections which arise from successive rescatterings.

Such an approach has been used extensively for a long time for studying the effects of pairwise final-state interactions[30]; but two essential questions remained to be answered. Do the amplitudes also have good properties with respect to the three-body squared energy variable? Can they be used in the neighborhood of a three-body resonance where three-body unitarity is required to be nearly fulfilled and where the successive rescattering corrections are certainly important?[31] We see that three-body unitarity has to be checked *a posteriori* in this approach, in contradistinction to what occurs in other mass-shell (and probably more powerful) S-matrix approaches[32] in which three-body unitarity is also an input.

We have examined this question by retaining, for the sake of simplicity, only one partial wave in two-body unitarity. Such an approach, as well, indeed, as any limitation on the number of the subenergy partial waves, suffers from the well-known drawbacks of a Cini-Fubini-type approximation,[20,33] but an important advantage is that it is simple enough to allow the determination of the complete analytic structure of the amplitudes. In particular, the analytic properties in the total squared energy variable, as well as the precise form of the associated three-body discontinuity (especially the constraints imposed to the subenergy path of integration), are all generated by the model itself.

In both cases of angular momentum which we have considered, the same procedure has been applied for deriving these properties from the integral equations. Nevertheless, there is a difference. In the case $J=0$, $l=0$ the study is well supported by the comparison with perturbation theory, and the kernels of the equations are simply related to the Feynman triangle graph with three scalar internal particles. No similar connection can be made in the case $J=1$, $l=1$ since the associated Feynman triangle graph with two scalar and one spin $l=1$ internal particles diverges.[34]

In both cases, however, we have obtained amplitudes having approximately the same properties when the singularities due to the two-body forces are neglected (i.e. $n=N/q_{2c}^{2l}=$const[35] for $l=0, 1$): The amplitudes have the same form as the usual isobaric amplitudes (i.e., their angular dependences appear explicitly) they can nearly satisfy three-body unitarity in many practical cases.

This last property requires the $3\rightarrow 3$ amplitudes involved in the model to be nearly symmetric with respect to the initial and final variables. This may be satisfied if (a) either the integral part of the equations is small compared to the symmetric inhomogeneous term—this

[29] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[30] I. J. R. Aitchison, Nuovo Cimento **51A**, 249 (1967), and references therein.

[31] Such questions have been first examined by G. Bonnevay, in *Proceedings of the Tenth Annual International Conference on High Energy Physics, Rochester, 1960*, edited by E. C. G. Sudarshan (Interscience Publishers, Inc., New York, 1961); Nuovo Cimento **30**, 1325 (1963).

[32] S. Mandelstam, Phys. Rev. **140**, B375 (1965).

[33] Froissart bounds in particular may only be satisfied in the two cases of angular momentum $l=0$, 1 that we have considered, but not for $l>1$.

[34] From the integral equations derived in I, it is easy to extract an equivalent trianglelike amplitude. However, we have not studied in detail the further relations which may hold between this amplitude and the true divergent Feynman graph.

[35] Indeed, the same result certainly remains valid for nonconstant but holomorphic functions n, provided that suitable subtractions are done.

occurs in particular in the weak-coupling case in which the first iterations (especially the inhomogeneous term) dominate; or (b) the integral term is important but contributes essentially by its symmetric part. As noticed in I, this occurs, for instance, if D describes a sufficiently sharp two-body resonance.

Correspondingly, again when the singularities due to the two-body forces are neglected, it has been possible by truncating the integral equations to get amplitudes satisfying three-body unitarity rigorously, at least in the decay region. This procedure thus exhibits a region in which two-body unitarity, together with analyticity and a form of crossing, generates three-body unitarity. Because an inverse statement also holds,[27] we would like to claim here once again[36] that under the preceding approximations there is an apparent connection between three basic inputs of S-matrix theory: unitarity, analyticity, and crossing.

Another interesting alternative which can be looked for within the present model consists in working with nonconstant n functions. This allows better convergence properties for the nontruncated equations which, for $n=$const. are of the Fredholm type only if at infinity $D(s)=0(s^{l+\epsilon})$, $\epsilon>0$. As regards three-body unitarity, however, one can only claim qualitative predictions, which must be tested on a computer. This numerical aspect is under investigation in the particular case of the three-pion system.

APPENDIX A: KINEMATICAL NOTATIONS

We have collected in this Appendix many notations and formulas which would have overburdened the text. For reasons of simplicity, the variable s, which only plays the role of a parameter, is often omitted. The first formulas concern the magnitudes of the five momenta involved in Fig. 1. They may be expressed in terms of

$$k^2(a^2,b^2,c^2)=[a^2-(b-c)^2][a^2-(b+c)^2],$$

and, more precisely, ($i=1, 2, 3$)

$$k_i=k(s_i,s,1)$$
$$K_i=k(s_i,1,1)$$
$$k_s=k(s,1,1).$$

We specify k_i, K_i, and k_s as positive in the decay or production region; this region, as well as the other physical regions associated with the quasi-four-leg processes of Fig. 1, are delimited by the Kibble curve[13]

$$\varphi(s_1,s_2)\equiv s_1s_2s_3-(s-1)^2=0,$$

with $s_1+s_2+s_3=s+3$; φ and $\sqrt{\varphi}$ are positive in all the physical regions.

The trigonometrical lines of the angles used in the text may be expressed in terms of these quantities. First, the scattering angle of the $2\to 2$ reaction [Fig. 1(b)] is θ_{12}, and

$$\sin\theta_{12}=2[s_1\varphi(s_1,s_2)]^{1/2}/k_1K_1,$$
$$\cos\theta_{12}=2s_1\gamma(s_1,s_2)/k_1K_1$$

with

$$\gamma(s_1,s_2)=s_2-\tfrac{1}{2}(s+3-s_1)=\tfrac{1}{2}(s_2-s_3).$$

The angle χ_{21} of the crossing relations Eq. (2.7) is such that

$$\sin\chi_{21}=-2[s\varphi(s_1,s_2)]^{1/2}/k_1k_2,$$
$$\cos\chi_{21}=N_\chi(s_1,s_2)/k_1k_2,$$

with

$$N_\chi(s_1,s_2)=(s_1+s-1)(s_2+s-1)+2s(1-s).$$

We also need some trigonometrical lines of the angles α_i and β_i [see Fig. 2(a) and Eqs. (2.16) and (2.17)]. It is sufficient to know the following relations:

$$\sin\alpha_1\cos\beta_1=\frac{-\cos\alpha_2+\cos\alpha_1\cos\chi_{21}}{\sin\chi_{21}}$$

(and similar relations obtained by a cyclic permutation over the indices), and

$$\cos\alpha_i=\mathfrak{N}(\alpha_i)/k_sk_i,\quad i=1,2,3$$

where $\mathfrak{N}(\alpha_i)$ is a polynomial with respect to the invariants [the dependence is more precisely linear as regards the momentum transfers of Fig. 1(a)].

From these results, it is possible to construct explicitly the matrix $\bar{M}$ of Eq. (2.11) and then to evaluate the functions $\beta_{J\Lambda_1}{}^{(\eta)}$ of Eq. (2.13). These, for $J=1$, read[12]

$$\beta_{11(1)}{}^{(+)}(s_1,s_2)=[\varphi(s_1,s_2)]^{1/2},$$
$$\beta_{11(1)}{}^{(-)}(s_1,s_2)=[\varphi(s_1,s_2)]^{1/2}/k_1,\quad \beta_{10(1)}{}^{(-)}(s_1,s_2)=1/k_1.$$

APPENDIX B: SYMMETRY OF KERNELS

We give in this Appendix the kernels involved in the integral equations (3.12) and (3.14). They can be split into two parts, in the same manner as the kernel of Eq. (C1) in paper I. The first part is explicitly given by the formulas

$\eta=+1$:

$$^1\Delta^1(s_2',s,s_1)=\frac{1}{\pi k_2'^3}\int_{s_-(s,s_2')}^{s_+(s,s_2')}\frac{ds_1'}{s_1'-s_1}\frac{n(s_1')}{k_1'^3}\times{}^1C^1(s_2')\,{}^1P^1(s_2',s_1'),\quad \text{(B1)}$$

with

$$^1P^1(s_2',s_1')=\varphi(s_1',s_2'),\quad {}^1C^1(s_2')=\tfrac{3}{2}k_2'^3;\quad \text{(B2)}$$

$\eta=-1$:

$$^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)=\frac{1}{v}\begin{vmatrix}{}^{\Lambda_2}\delta_+(s_2',s,s_1) & u_+{}^{\Lambda_1}\\ {}^{\Lambda_2}\delta_-(s_2',s,s_1) & u_-{}^{\Lambda_1}\end{vmatrix},\quad \text{(B3)}$$

[36] I. J. R. Aitchison and R. Pasquier, Phys. Rev. 152, 1274 (1966).

where

$$^{\Lambda_2}\delta_{\pm}(s_2',s,s_1)=\frac{s_1-(\sqrt{s}\pm1)^2}{\pi k_2'^3}\int_{s_-(s,s_2')}^{s_+(s,s_2')}\frac{ds_1'}{s_1'-s_1}\frac{n(s_1')}{k_1'^3}$$

$$\times\frac{-u_{\pm}{}^1\,{}^{\Lambda_2}C^0(s_2')\,{}^{\Lambda_2}P^0(s_2',s_1')+u_{\pm}{}^0\,{}^{\Lambda_2}C^1(s_2')\,{}^{\Lambda_2}P^1(s_2',s_1')}{s_1'-(\sqrt{s}\pm1)^2},$$

$$u_+{}^0=u_-{}^0=\sqrt{2},\quad u_+{}^1=-1-\sqrt{s},\quad u_-{}^1=+1-\sqrt{s},$$

$$v=u_+{}^0u_-{}^1-u_-{}^0u_+{}^1=2\sqrt{2}.$$

Notice that the $u_{\pm}{}^{\Lambda_1}$ are nothing but the coefficients of f^0 and f^1 in the kinematical constraint relations [Eq. (3.10)]

$$^0P^0(s_2',s_1')=N_\chi(s_1',s_2')\gamma(s_1',s_2')\gamma(s_2',s_1'),$$
$$^0C^0(s_2')=-3s_2'k_2',$$
$$^0P^1(s_2',s_1')=\varphi(s_1',s_2')\gamma(s_2',s_1'),$$
$$^0C^1(s_2')=\tfrac{3}{2}\sqrt{2}(\sqrt{s})s_2'k_2',$$
$$^1P^0(s_2',s_1')=\varphi(s_1',s_2')\gamma(s_1',s_2'),$$
$$^1C^0(s_2')=3\times2\times(\sqrt{2}\sqrt{s})k_2',\quad\text{(B4)}$$
$$^1P^1(s_2',s_1')=\varphi(s_1',s_2')N_\chi(s_1',s_2'),$$
$$^1C^1(s_2')=\tfrac{3}{2}k_2'.$$

n is $N/q_{2c}{}^2$, N being the N function of the $2\rightarrow2$ scattering matrix with $l=1$, and $q_{2c}{}^2$ is the squared momentum of particles (2) and (3) in the s_1 c.m. system (i.e., the centrifugal factor for $l=1$). The kinematical functions φ, N_χ, and γ are given explicitly in Appendix A.

The second part $^{\Lambda_2}\mathrm{K}^{\Lambda_1}(s_2',s,s_1)$ of the kernel follows from the preceding $^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)$ by replacing the integration path $[s_-(s,s_2'),\,s_+(s,s_2')]$ by $[\tilde{s}(s,s_2'),\,+\infty]$ just as in I (the precise meaning and the possible distortions of these contours can be specified as in that paper).

When the effects due to the two-body forces are neglected (i.e., $n=1$) the parts $^{\Lambda_2}\Delta^{\Lambda_1}$ possess particular properties of symmetry. As in I, it is convenient to deduce them from a comparison with the J projection ($J=1$ in the present case) of a well-defined OPE amplitude.

The OPE process which we consider is illustrated in Fig. 3. Each bubble function which depends on t'', is assumed to be well represented by the $l=1$ centrifugal factor only. The associated amplitude thus takes the form

$$\mathbf{Q}(q',q)=\frac{1}{\pi}\frac{3K_1k(s_1,t'',1)P_1(\hat{z}_2)}{4s_1}\frac{1}{t''-1}$$
$$\times\frac{3K_2'k(s_2',t'',1)P_1(\hat{z}_1)}{4s_2'}.\quad\text{(B5)}$$

The functions k and K_i are defined in Appendix A and the cosines $\hat{z}_i$ are associated with the invariants $\hat{s}_i$ in Fig. 3.

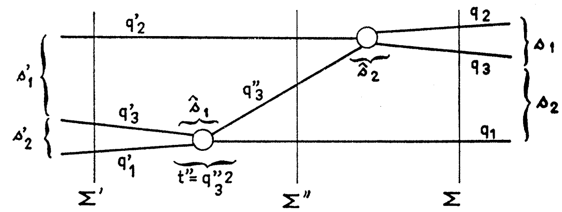

FIG. 3. The OPE process considered in Appendix B.

The $J=1$ projections of $\mathbf{Q}(q',q)$ that it is convenient to consider are

$$^{\Lambda_2}\mathbf{Q}^{\Lambda_1}(s_2',s_3',s,s_1,s_2)=\frac{1}{8\pi^2}\int\mathbf{Q}(q',q)$$
$$\times\mathfrak{D}_{\Lambda_1\Lambda_2}{}^1[\mathcal{E}^{-1}(\mathbf{q}_1)\mathcal{E}(\mathbf{q}_2')]d[\mathcal{E}^{-1}(\mathbf{q}_1)\mathcal{E}(\mathbf{q}_2')],$$

where $\mathbf{q}_1$ and $\mathbf{q}_2'$ are taken as privileged momenta for the final and initial states, respectively [for choice (h) of axes, see Sec. II]. After some straightforward manipulations, they may be rewritten as

$$^{\Lambda_2}\mathbf{Q}^{\Lambda_1}(s_2',s_3',s,s_1,s_2)=\frac{3^2}{2}\frac{K_1K_2'}{16s_1s_2'}d_{\Lambda_10}{}^1(\theta_{12})d_{\Lambda_20}{}^1(\theta_{23}')$$
$$\times\frac{2s}{k_1k_2'}\int_{t_+}^{t_-}\frac{dt''}{t''-1}k(s_1,t'',1)k(s_2',t'',1)d_{\Lambda_2\Lambda_1}{}^1(\chi_{21}'')$$
$$\times d_{\Lambda_10}{}^1(\theta_{12}'')d_{\Lambda_20}{}^1(\theta_{23}''),\quad\text{(B6)}$$

with

$$t_{\pm}=\tfrac{1}{2}[s_1+s_2'+2-s+(1-s_1)(1-s_2')/s\pm k_1k_2'/s].$$

The definition of the angles (name and indices) is the same as in Appendix A, but here one mass is equal to $\sqrt{t''}$ instead of 1. By convention, the number of primes on these angles refers to the section Σ, Σ', or Σ'' of Fig. 3 to which the two associated momenta belong.

Of course, these projected amplitudes are not free from kinematical singularities, but we may apply the same methods as in Sec. II and define regularized amplitudes by factoring $\beta_{1\Lambda_i}{}^{(\eta)}$, $i=1$, 2, out of suitable combinations of them imposed by parity conservation. Finally, as in Eq. (3.6) we are led to consider

$$^{\Lambda_2}G^{\Lambda_1}(s_2',s,s_1)={}^{\Lambda_2}\mathcal{G}^{\Lambda_1}/[\beta_{1\Lambda_1}{}^{(\eta)}(s_1,s_2)\beta_{1\Lambda_2}{}^{(\eta)}(s_2',s_3')$$
$$\times\ \xi_{\Lambda_1}(s_1,s_2)\xi_{\Lambda_2}(s_2',s_3')],\quad\text{(B7)}$$

where

$$^{\Lambda_2}\mathcal{G}^{\Lambda_1}=\tfrac{1}{2}[{}^{\Lambda_2}\mathbf{Q}^{\Lambda_1}+\eta(-)^{1+\Lambda_1}\,{}^{\Lambda_2}\mathbf{Q}^{-\Lambda_1}]$$
$$=\tfrac{1}{2}[{}^{\Lambda_2}\mathbf{Q}^{\Lambda_1}+\eta(-)^{1+\Lambda_2}\,{}^{-\Lambda_2}\mathbf{Q}^{\Lambda_1}],$$

with η defined as in Sec. II.

Indeed, it is more convenient to work with amplitudes $^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)$ free from trivial numerical factors present in the $^{\Lambda_2}G^{\Lambda_1}(s_2',s,s_1)$. The $^{\Lambda_2}\tilde{G}^{\Lambda_1}$ are represented by

$$^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)=\frac{1}{\pi k_1{}^3k_2'^3}\int_{t_+}^{t_-}{}^{\Lambda_2}P^{\Lambda_1}(s_2',s_1,t'')\frac{dt''}{t''-1},\quad\text{(B8)}$$

where the functions ${}^{\Lambda_2}P^{\Lambda_1}(s_2',s_1,t'')$ can be deduced from Eqs. (B6) and (B7) and for $t''=1$ reduce to the polynomials ${}^{\Lambda_2}P^{\Lambda_1}(s_2,s_1)$ of Eqs. (B2) and (B4).

For $\eta=-1$, these amplitudes ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)$ can possess poles at $s_1=0$ and/or $s_2'=0$, have branch points at $t_\pm=1$, i.e., $s_1=s_\pm(s,s_2')$ [despite the presence of the factors $1/k_1^3k_2'^3$, the points $s_1=(\sqrt{s}\pm1)^2$ and $s_2'=(\sqrt{s}\pm1)^2$ are not singular], and behave at infinity in s_1 (or s_2') like a constant. They can thus be rewritten in terms of dispersion integrals in the s_1 plane with a subtraction which is convenient to do at $s_1=(\sqrt{s}+1)^2$ or $s_1=(\sqrt{s}-1)^2$. We thus get

$$\begin{aligned}{}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)=&\frac{|s_1\,{}^{\Lambda_2}\tilde{G}^{\Lambda_1}|_{s_1=0}}{-(\sqrt{s}\pm1)^2s_1}[s_1-(\sqrt{s}\pm1)^2]\\&+{}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,(\sqrt{s}\pm1)^2)\\&+\frac{s_1-(\sqrt{s}\pm1)^2}{\pi k_2'^3}\int_{s_-(s,s_2')}^{s_+(s,s_2')}{}^{\Lambda_2}P^{\Lambda_1}(s_2',s_1')\\&\times\frac{1}{k_1'^3}\frac{ds_1'}{(s_1'-s_1)[s_1'-(\sqrt{s}\pm1)^2]},\end{aligned}\qquad\text{(B9)}$$

wehre $|s_1{}^{\Lambda_2}\tilde{G}^{\Lambda_1}|_{s_1=0}$ stands for the residue of ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)$ at $s_1=0$, if any, and can be easily found from Eq. (B8) (indeed, only ${}^0\tilde{G}^0$ and ${}^1\tilde{G}^0$ have a pole at $s_1=0$). The precise meaning of the s_1' path of integration is unambiguously determined once the t'' path in Eq. (B6) is given. As one can verify,[36,37] the $-i\epsilon$ (physical) prescription for the internal mass of the OPE process (Fig. 3) leads in particular to the same s_1' contours as those involved in the integral equations (3.12) for the physical amplitude f^Λ (i.e., the amplitude f^Λ obtained by leaving the variables s and s_i to reach their respective cuts from above).

Now, by suitable combinations of the ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}$ given by Eq. (B9), it is possible to reconstruct kernels ${}^{\Lambda_2}\tilde{\Delta}^{\Lambda_1}$ defined through Eq. (B3) and through:

$${}^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)={}^{\Lambda_2}C^{\Lambda_1}(s_2')\,{}^{\Lambda_2}\tilde{\Delta}^{\Lambda_1}(s_2',s,s_1)\,.\qquad\text{(B10)}$$

It happens that these combinations are of the "kinematical constraints" type and such that the terms ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,(\sqrt{s}\pm1)^2)$ cancel. The remaining terms, which alone in the end prevent the ${}^{\Lambda_2}\tilde{\Delta}^{\Lambda_1}(s_2',s,s_1)$ and the ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)$ from being equal, originate from the presence of poles at $s_1=0$. The important point is that these supplementary terms have good symmetry properties with respect to the initial and final subenergy variables, and themselves satisfy the kinematical constraints. As one can verify, these two properties also hold for the ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}(s_2',s,s_1)$ [the symmetry of the ${}^{\Lambda_2}\tilde{G}^{\Lambda_1}$ clearly appears in the integral form (B6) or (B8)]; the same is then also true for the ${}^{\Lambda_2}\tilde{\Delta}^{\Lambda_1}(s_2',s,s_1)$. As a consequence, the ${}^{\Lambda_2}\Delta^{\Lambda_1}(s_2',s,s_1)$ are polar kernels; incidentally, we also find that they satisfy the kinematical constraints (3.10) in s_1, but this property already follows from the derivation of the integral equation (3.12).

[37] R. Pasquier, Orsay Report No. IPNO/TH 31, 1965 (unpublished).

In the case $\eta=+1$, the symmetry of the kernel ${}^1\tilde{\Delta}^1(s_2',s,s_1)$ can be derived in a similar and much simpler manner: then ${}^1\tilde{G}^1(s_2',s,s_1)$ has no pole in $s_1=0$ (or $s_2'=0$), is $O(1/s_1)$ [or $O(1/s_2')$] at infinity in s_1 (or s_2'), and can thus be identified with ${}^1\tilde{\Delta}^1(s_2',s,s_1)$ [Eq. (B10)] by an unsubtracted dispersion relation in the s_1 plane. The symmetry of ${}^1\tilde{\Delta}^1(s_2',s,s_1)$ then follows from that of ${}^1\tilde{G}^1(s_2',s,s_1)$.

APPENDIX C: ISOSPIN AND BOSE-EINSTEIN STATISTICS

In the present model, isospin may as well be introduced by considering the quasi-four-leg process of Fig. 1 as a decay [Fig. 1(a)] or a scattering [Fig. 1(b)].

In the context of Fig. 1(a), we are dealing with a pseudoparticle[38] of isospin $\mathcal{T}$ and charge $\mathcal{T}_z$ decaying into three particles of isospin $t=1$ and charge $t_z=\tau$. As usual, the total isospin states $|\mathcal{T}\mathcal{T}_z\rangle$ may be obtained by first coupling the isospins of a pair of final particles and then the result to the remaining isospin. According to the choice of the final pair, different decompositions may be obtained, and crossing relations may be written between the isospin amplitudes. These considerations are rather analogous to what we have encountered in Secs. II and III for the spin projections; the role of Λ_i is here played by the total isospin T_i of a pair of particles.

The consideration of Fig. 1(b), however, allows a more natural introduction of charge-independent amplitudes. This procedure, moreover, uses more common developments and notations. We are first led to consider the scattering matrix elements obtained from the decay matrix elements by crossing particle (i); according to the usual phase convention,[39,40] we have

$$\langle\mathcal{T}\mathcal{T}_zt_i-\tau_i|\mathbf{R}|t_j\tau_jt_k\tau_k\rangle=(-)^{\tau_i}\langle\mathcal{T}\mathcal{T}_z|\mathbf{R}|t_i\tau_it_j\tau_jt_k\tau_k\rangle,\qquad\text{(C1)}$$

where (i,j,k) stands for a cyclic permutation of (1,2,3). These scattering matrix elements may then be expanded over both the initial and final isospin states, in order to exhibit charge-independent amplitudes. This gives

$$\langle\mathcal{T}\mathcal{T}_zt_i-\tau_i|\mathbf{R}|t_j\tau_jt_k\tau_k\rangle=\sum_{T_iT_{iz}}\langle\mathcal{T}t_i\mathcal{T}_z-\tau_i|T_iT_{iz}\rangle\mathbf{R}^{T_i}\langle T_iT_{iz}|t_j\tau_jt_k\tau_k\rangle,\qquad\text{(C2)}$$

where T_i and T_{iz} denote the isospin and charge of the pair (j,k) of final particles. Of course, depending on

[38] We do not consider the isospins of the initial particles in the case of a production, but only the resulting isospin state $|\mathcal{T}\mathcal{T}_z\rangle$.

[39] de Swart, Rev. Mod. Phys. 35, 916 (1963).

[40] B. Diu, in *Methods in Subnuclear Physics*, edited by M. Nikolić (Gordon and Breach Science Publishers, Inc., New York, 1968), p. 143.

which pair (j,k) is considered, different decompositions of $\mathbf{R}$ are obtained; by comparing them, one may write crossing relations such as

$$\mathbf{R}^{T_1}=\sum_{T_2} C_{T_1T_2}\mathbf{R}^{T_2}, \tag{C3}$$

with $C_{T_1T_2}$ as in Ref. (40). Here again, this condition can be fulfilled by imposing it upon each component in Eq. (3.2′); this gives

$$\begin{aligned}\mathbf{R}^{T_1}&=\mathrm{R}_0{}^{T_1}+\mathrm{R}_1{}^{T_1}+\mathrm{R}_2{}^{T_1}+\mathrm{R}_3{}^{T_1}\\ &=\mathrm{R}_0{}^{T_1}+\mathrm{R}_1{}^{T_1}+\sum_{T_2} C_{T_1T_2}\mathrm{R}_2{}^{T_2}+\sum_{T_3} C_{T_1T_3}\mathrm{R}_3{}^{T_3}.\end{aligned} \tag{C4}$$

This development, as well as Eqs. (C2), (2.2), and (2.9), allows us now to examine whether in the model the full decay amplitude $\mathbf{R}$ is symmetric in the exchange of two final particles. We have, explicitly ,

$$\begin{aligned}\langle p_1p_2\mathcal{T}\mathcal{T}_z|\mathbf{R}|q_1q_2q_3\tau_1\tau_2\tau_3\rangle &=\sum_i \sum_{J\Lambda_iT_iT_{iz}l_i} \mathfrak{D}_{\Lambda_i0}{}^{J*}(\beta_i,\alpha_i,0)d_{\Lambda_i0}{}^{l_i}(\theta_{ij})\\ &\quad\times\mathrm{R}_i{}^{Jl_i\Lambda_iT_i}(s,s_i)(-)^{\tau_i}\langle\mathcal{T}t_i\mathcal{T}_z-\tau_i|T_iT_{iz}\rangle\\ &\quad\times\langle T_iT_{iz}|t_jt_k\tau_j\tau_k\rangle,\end{aligned} \tag{C5}$$

where (i,j,k) is again a cyclic permutation of (1,2,3). Such an amplitude remains invariant under the exchange of particles (2) and (3) if the term $i=1$ is invariant and if the terms $i=2$ and $i=3$ exchange with each other. Look first at $i=1$: under the transformation, β_1 increases or decreases by π, θ_{12} becomes $\pi-\theta_{13}$ and the second Clebsch-Gordan coefficient is multiplied by $(-)^{T_1}$; in sum, this term is invariant if it only contains summation over l_1 and T_1 of the same parity. Notice that this condition "$l+T$ even" is nothing but the familiar Bose-Einstein requirement in a two-body π-π channel. The exchange of the terms $i=2$ and and $i=3$ also implies a summation over l_i and T_i of the same parity but, in addition, requires the functions $\mathrm{R}_2{}^{Jl_2\Lambda_2T_2}(s,s_2)$ and $\mathrm{R}_3{}^{Jl_3\Lambda_3T_3}(s,s_3)$ to take the same values when the subenergies, J, l, Λ, and T are equal. Of course the exchange of two other particles, (1) and (3) for instance, gives a similar result and requires in particular the function $\mathrm{R}_1{}^{Jl_1\Lambda_1T_1}(s,s_1)$ to be the same as the two others.

It remains now to insert isospin into the integral equations of Sec. III. As one can verify, it is sufficient to label the amplitudes involved in these equations with an additional index of isospin T_i. In Eqs. (3.12) and (3.14) in particular, we are now dealing with the amplitudes $f_1{}^{\Lambda_1T_1}(s_1)$ on the left-hand side and $f_2{}^{\Lambda_2T_1}(s_2')$ and $f_3{}^{\Lambda_3T_1}(s_3')$ on the right-hand side (reinstate for a moment the indices 2 and 3 on f). Thanks to Eq. (C3) these last two amplitudes may be expressed in terms of the "diagonal" contributions $f_2{}^{\Lambda_2T_2}(s_2')$ and $f_3{}^{\Lambda_3T_3}(s_3')$, which must be equal if the argument and all the upper indices are the same. If we specialize to the three-pion case with $l_i=1$, only one isospin state ($T_i=1$) is allowed and we simply have

$$f_i{}^{\Lambda_iT_1}=C_{T_1T_i}f_i{}^{\Lambda_iT_i},\quad i=2,3$$

with $C_{T_1T_i}=1$, $-\frac{1}{2}$, and $-\frac{1}{2}$ for $\mathcal{T}=0$, 1, and 2, respectively. In this case, the kernels of the integral equations for the isospin amplitudes are those given in Appendix B multiplied by the factor $C_{T_1T_i}$.